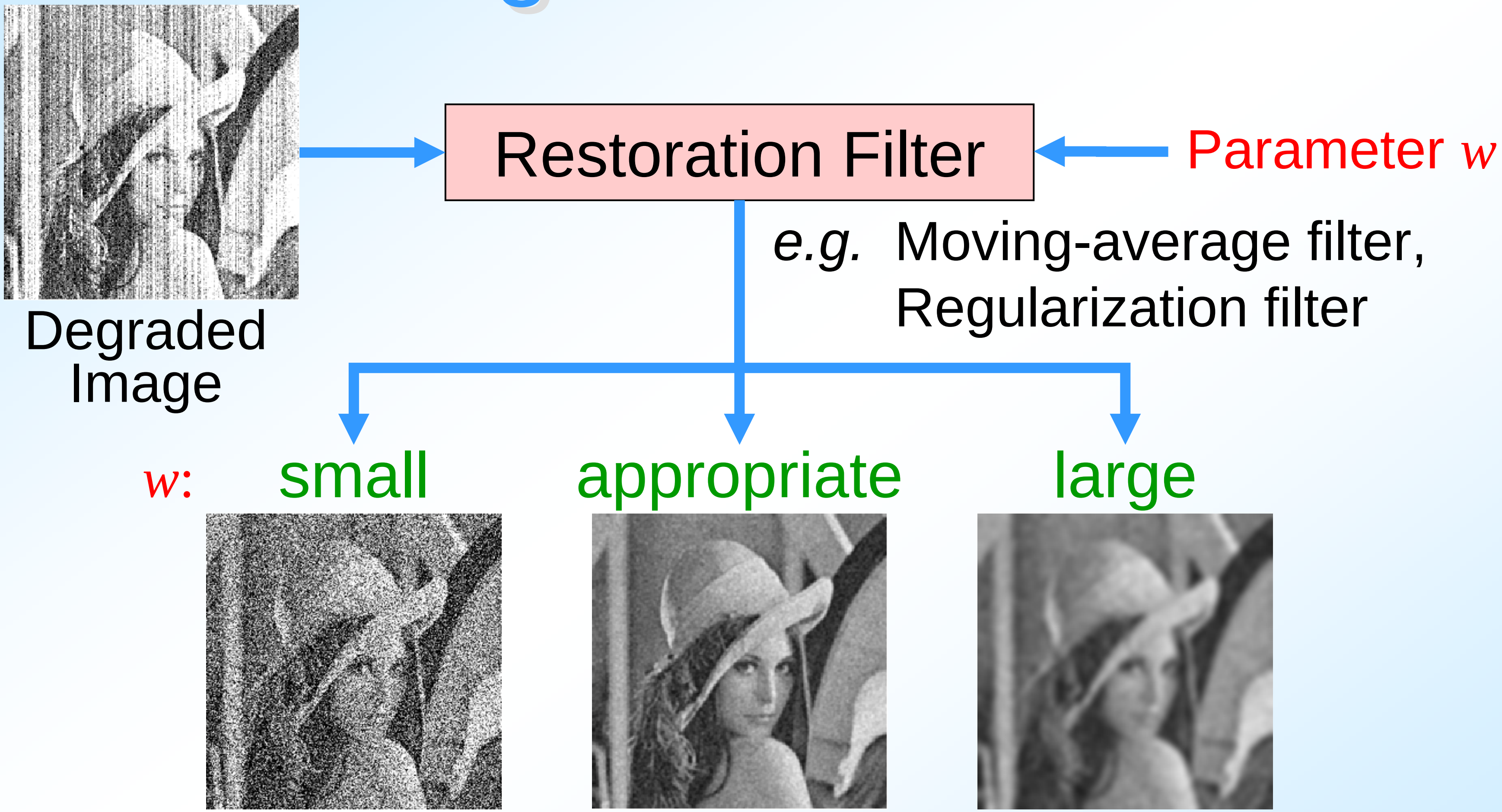
# Subspace Information Criterion for Image Restoration

## — Mean Squared Error Estimator for Linear Filters

Department of Computer Science,
Tokyo Institute of Technology, Japan.

Masashi Sugiyama
Hidemitsu Ogawa

# Image Restoration



Degraded Image

Restoration Filter

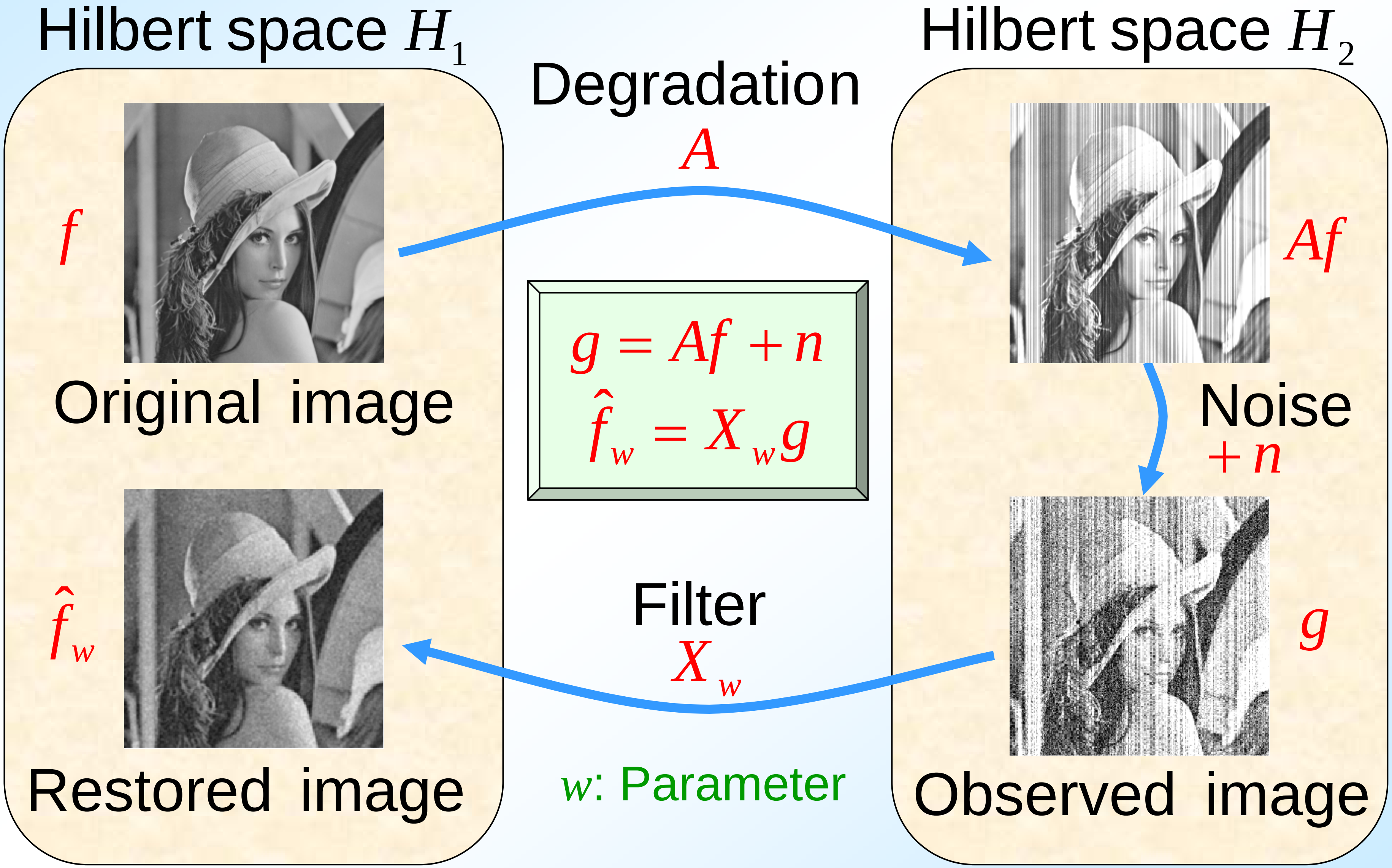
Parameter $w$

*e.g.* Moving-average filter,
Regularization filter

$w$: small appropriate large

We propose a method for determining parameter values appropriately.

# Formulation



Hilbert space $H_1$

Hilbert space $H_2$

Degradation

$A$

$f$

Original image

$Af$

$$g = Af + n$$

$$\hat{f}_w = X_w g$$

Noise

$+n$

$\hat{f}_w$

Restored image

Filter

$X_w$

$g$

Observed image

$w$: Parameter

# Goal of This Talk

We want to determine the parameter value $w$
so that Mean Squared Error (MSE) is minimized.

$$\mathrm{MSE} = \left\| \hat{f}_w - f \right\|^2$$

$\hat{f}_w$ : Restored image with parameter $w$

$f$ : Original image

However, it is impossible to directly calculate MSE since $f$ is unknown.

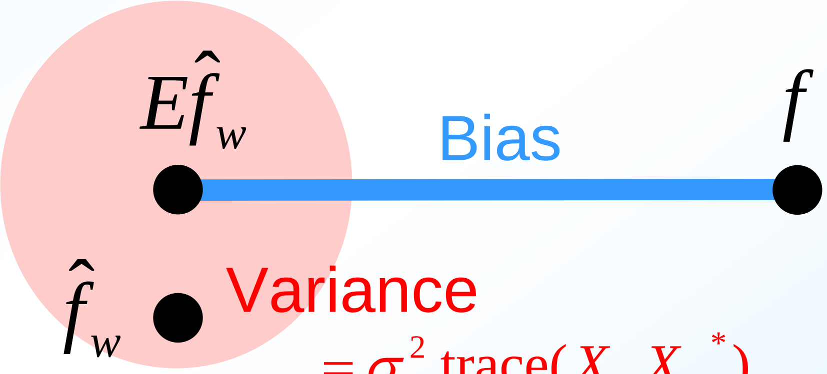
⬇

We propose an estimator of MSE called
the subspace information criterion (SIC),
and determine $w$ so that SIC is minimized.

# Bias / Variance Decomposition

$$E\,\text{MSE} = E\left\|\hat{f}_w - f\right\|^2$$

$$= \underbrace{\left\|E\hat{f}_w - f\right\|^2}_{\text{Bias}} + \underbrace{E\left\|\hat{f}_w - E\hat{f}_w\right\|^2}_{\text{Variance}}$$



$$= \sigma^2\ \text{trace}(X_w X_w^{\ *})$$

$E$ : Expectation over noise

$\hat{f}_w$ : Restored image

$f$ : Original image

$\sigma^2$ : Noise variance

$X_w$ : Restoration filter

$X_w^{\ *}$ : Adjoint of $X_w$

# Key Idea for Estimating Bias

$\hat{f}_u$ : Unbiased estimate of $f$ $(E\hat{f}_u = f)$.

$$\hat{f}_u = A^{-1}g$$

$$E\ A^{-1}g = A^{-1}Af + E\ A^{-1}n = f$$

$E$ : Expectation over noise

$f$ : Original image

$g$ : Degraded image

$A$ : Degradation operator

**POINT!**

$\hat{f}_u$ is used for estimating bias!!

# Bias Estimation

$$\text{Bias} = \left\| E\hat{f}_w - f \right\|^2$$

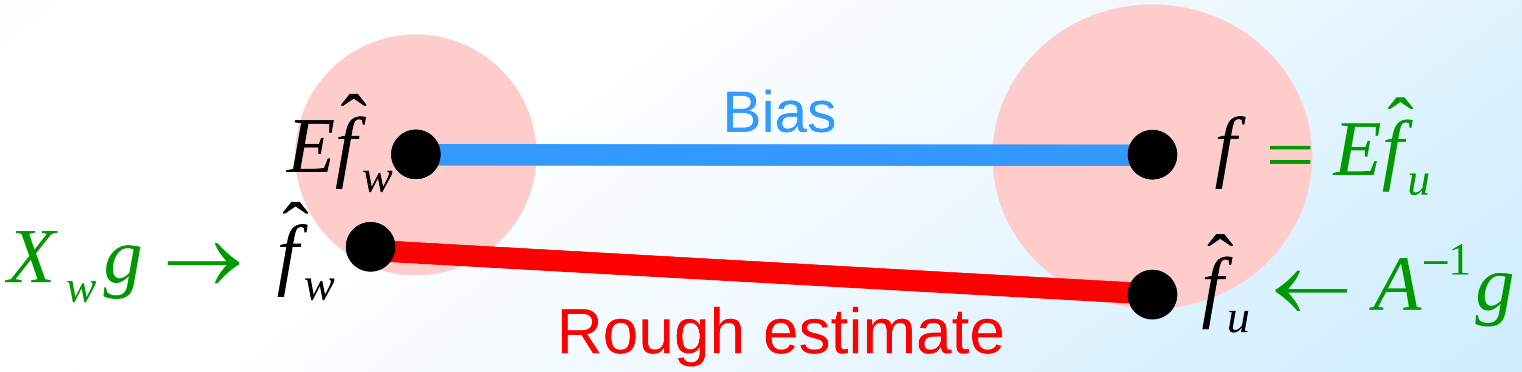
$$X_0 = X_w - A^{-1}$$

$$= \left\| \hat{f}_w - \hat{f}_u \right\|^2 - 2\left\langle X_0 A f, X_0 n \right\rangle - \left\| X_0 n \right\|^2$$

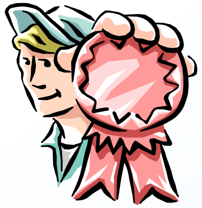
$$\Big\downarrow E \qquad \Big\downarrow E$$

$$\widehat{\text{Bias}} = \left\| \hat{f}_w - \hat{f}_u \right\|^2 - 0 - \sigma^2 \,\text{trace}\left(X_0 X_0^{\,*}\right)$$

$$\boxed{E\,\widehat{\text{Bias}} = \text{Bias}}$$



# Subspace Information Criterion (SIC)



$$\text{SIC} = \underbrace{\left\| \hat{f}_w - \hat{f}_u \right\|^2 - \sigma^2 \,\mathrm{trace}\left(X_0 X_0^*\right)}_{\text{Bias estimator}} + \underbrace{\sigma^2 \,\mathrm{trace}\left(X_w X_w^*\right)}_{\text{Variance}}$$

$$X_0 = X_w - A^{-1}$$

$\sigma^2$ : Noise variance

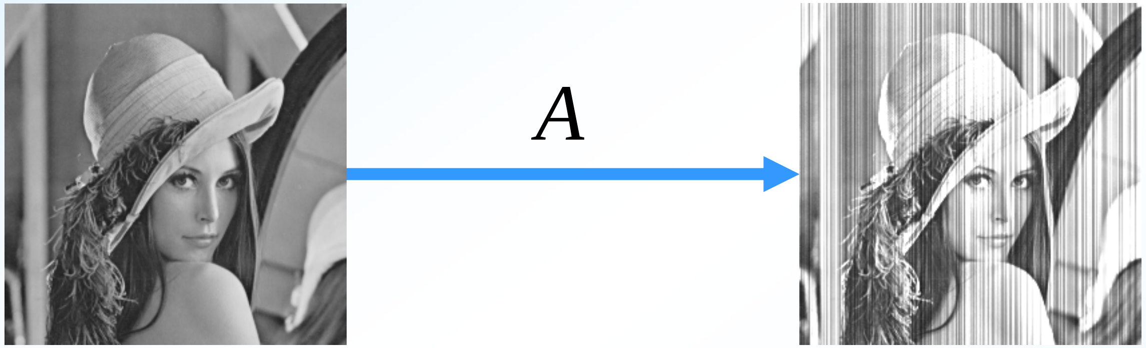
SIC is an unbiased estimator of expected MSE:

$$E\ \text{SIC} = E\ \text{MSE}$$

$$\text{MSE} = \left\| \hat{f}_w - f \right\|^2$$

# Simulation Setting

- Degradation operator $A$



$f(x,y)$ $[Af](x,y) = a_x f(x,y)$

$a_x$ : scaling factor

- Noise $n(x,y) \overset{i.i.d.}{\sim} N(0,\sigma^2)$

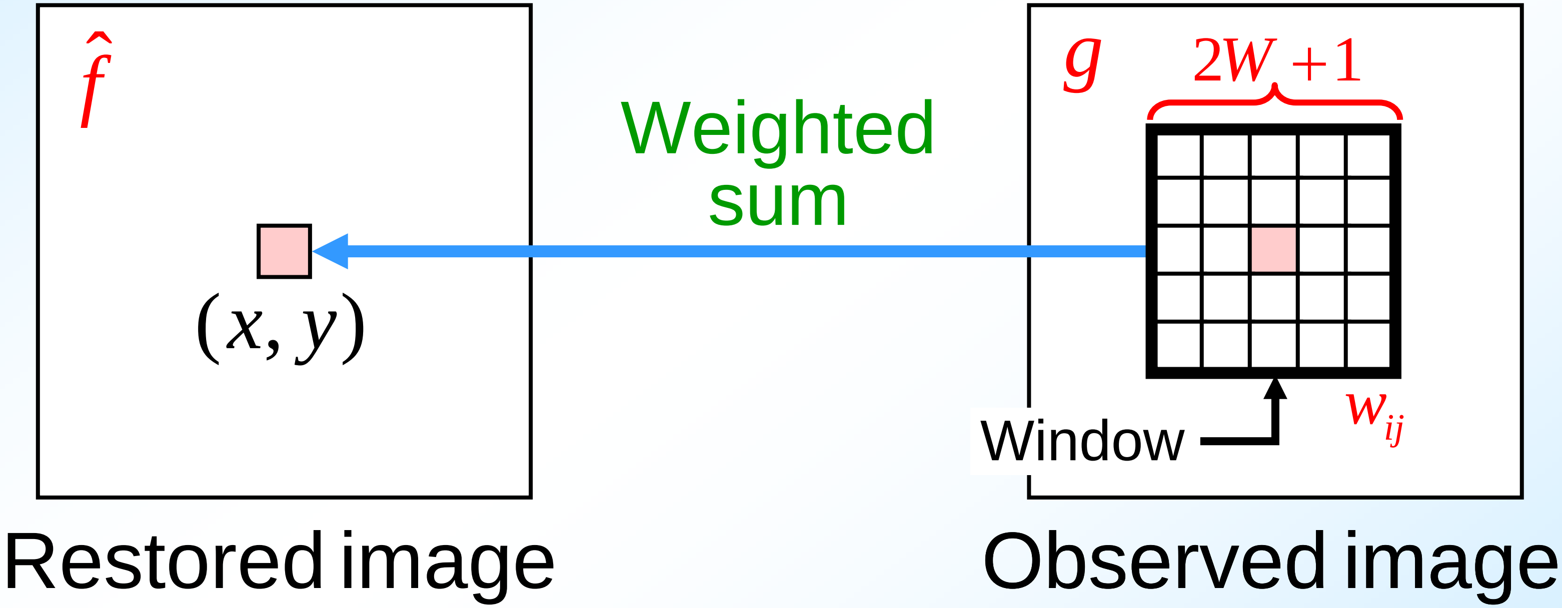
$\sigma^2 = 900$

- Filter $X = X'A^{-1}$

$X'$ : Moving-average filter

# Moving-Average Filter

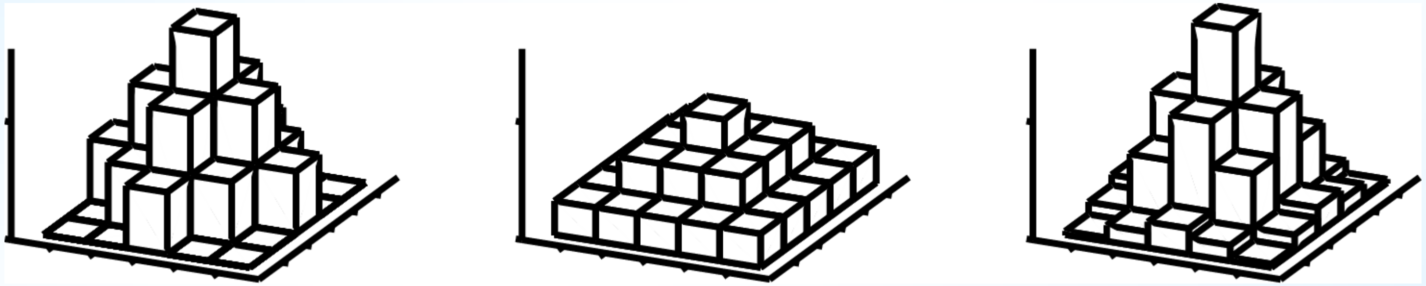
$$\hat{f}(x,y) = \sum_{i,j=-W}^{W} w_{ij}\, g(x-i, y-j)$$



Parameter : Window size $W$
Weight pattern $\{w_{ij}\}$

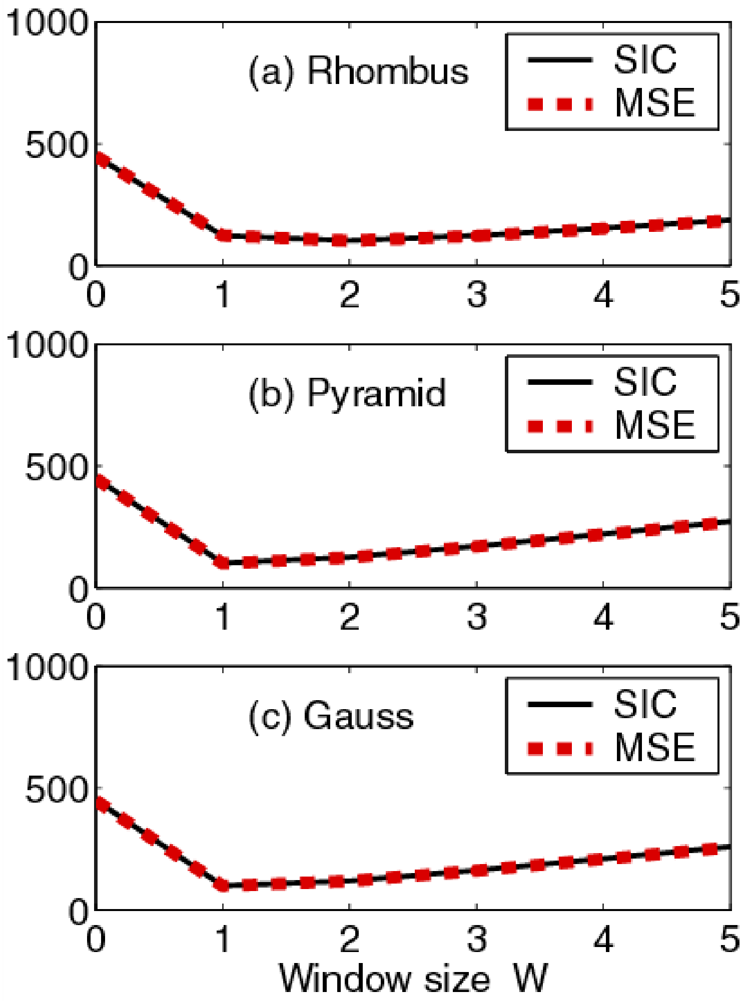
# Parameter candidates

- Window size $W = 0, 1, \ldots, 5$
- Weight pattern $\{w_{ij}\}$



(a) Rhombus (b) Pyramid (c) Gauss

# Simulation Results : Lena



$g$

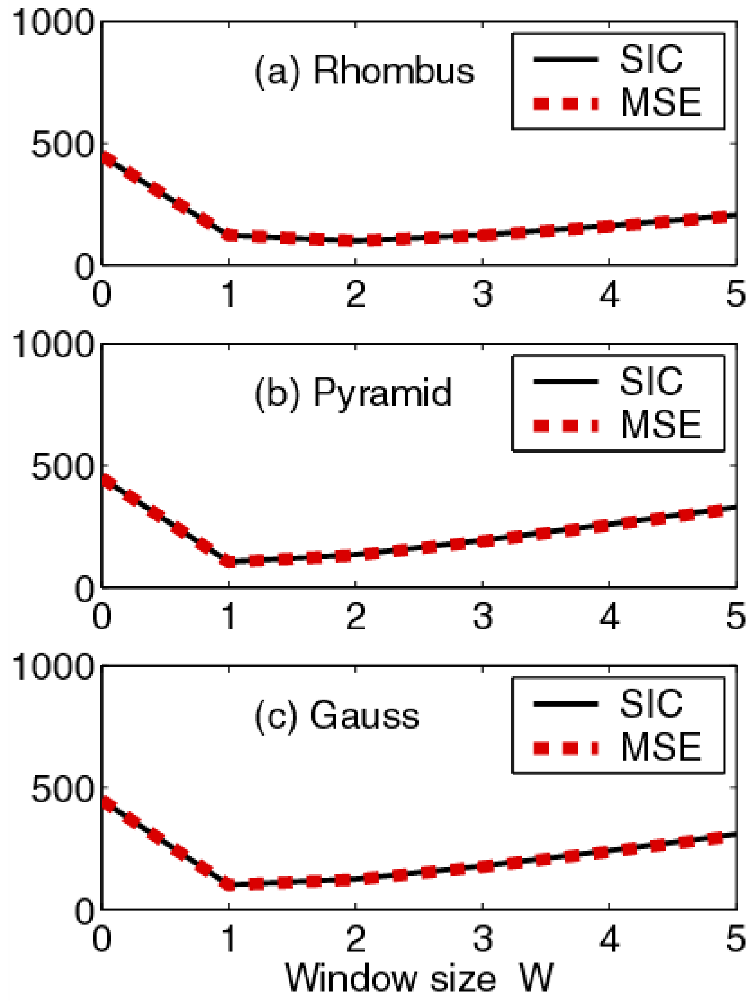
MSE=7046

$\hat{f}$

SIC: Gauss ($W$=1), MSE=99

OPT: Gauss ($W$=1)

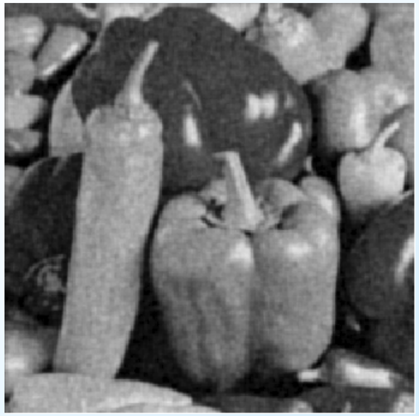
# Simulation Results : Peppers



$g$

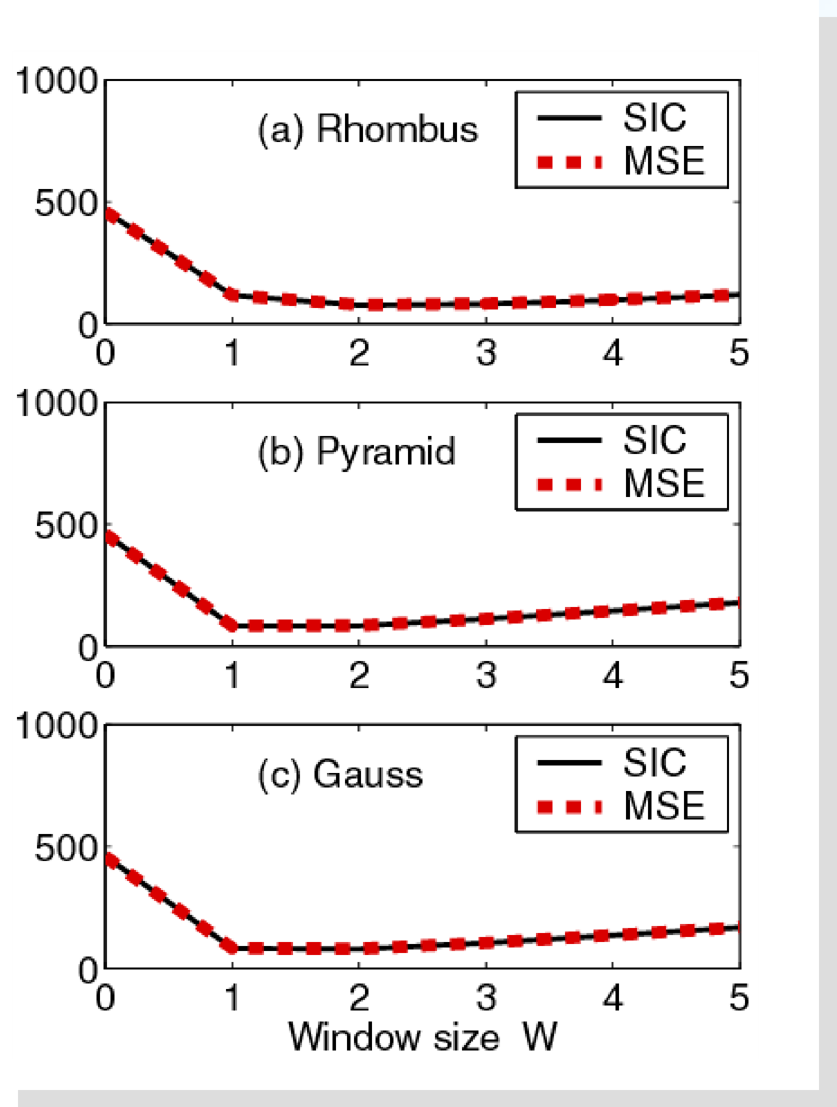
MSE=5572

$\hat{f}$



SIC: Rhombus ($W$=2), MSE=99
OPT: Rhombus ($W$=2)

# Simulation Results : Girl



$g$

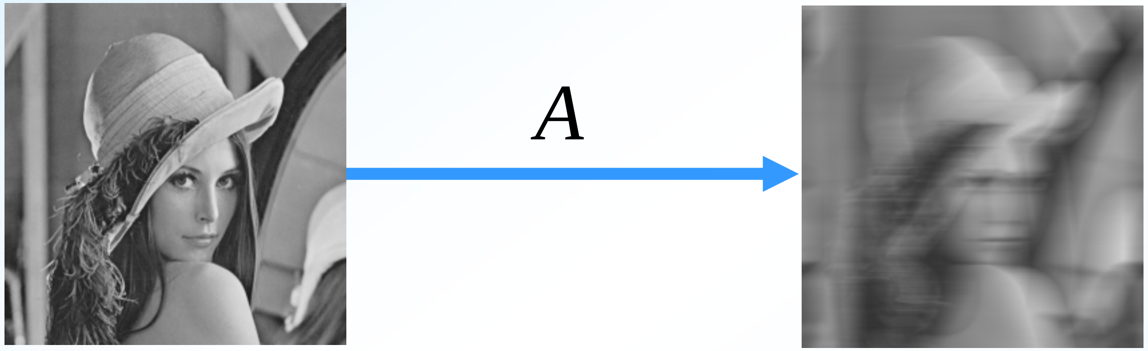
MSE=3217

$\hat{f}$

SIC: Rhombus ($W$=2), MSE=77
OPT: Rhombus ($W$=2)

# Simulation Setting (2)

- Degradation operator $A$



$f(x,y)$

$[Af](x,y)=\frac{1}{15}\sum_{i=-7}^{7} f(x-i,y)$

- Noise $n(x,y) \overset{i.i.d.}{\sim} N(0,\sigma^2)$

$\sigma^2 = 16$

- Filter: Regularization filter $X_\alpha$

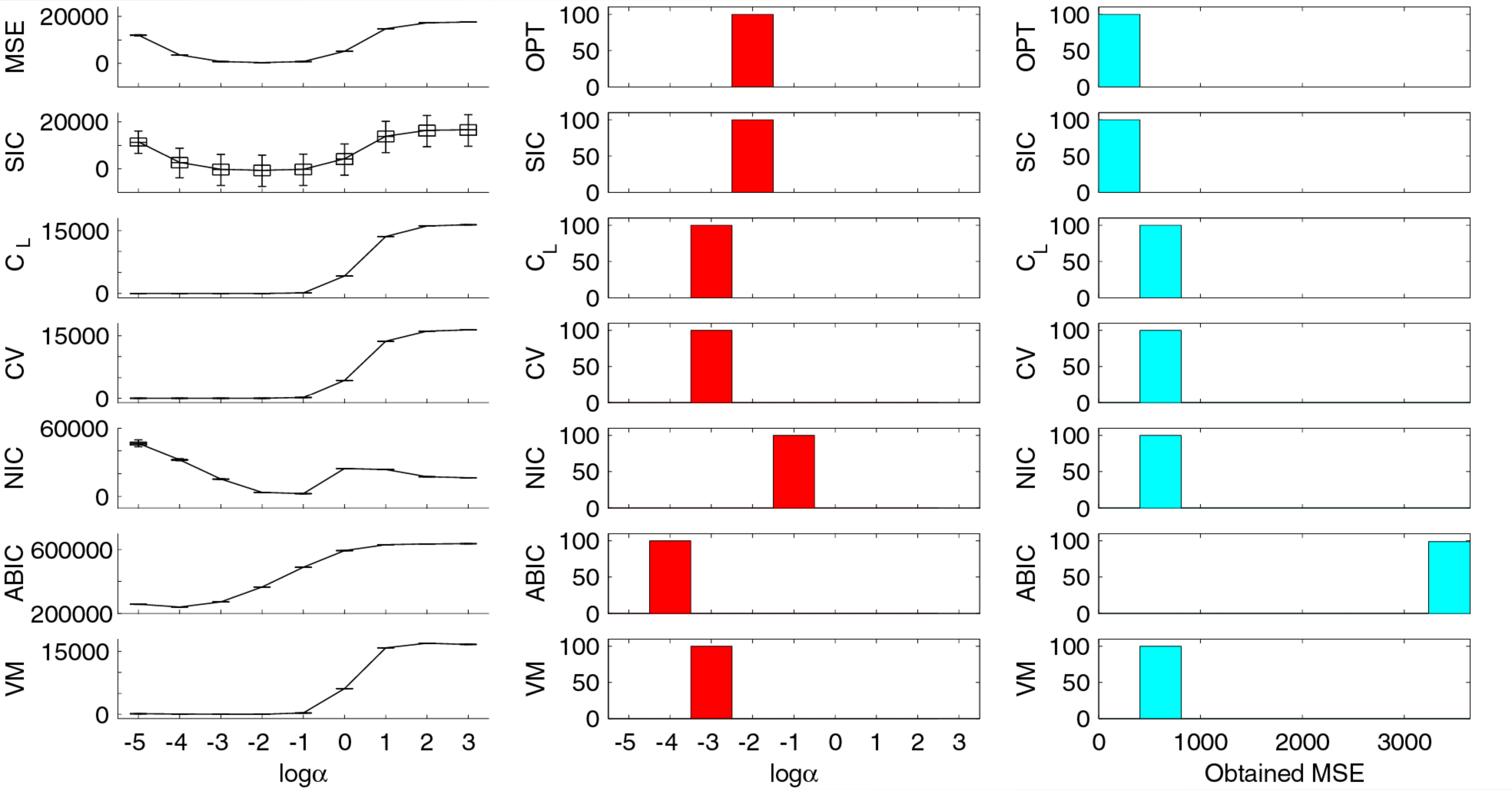
$\alpha$ : Regularization parameter

selected from $\{10^{-5},10^{-4},\ldots,10^{3}\}$

# Compared Methods

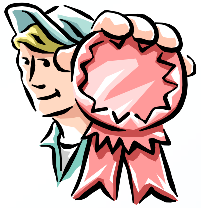
- Subspace information criterion (SIC)
- Mallows’s $C_L$ ($C_L$)
- Leave-one-out cross-validation (CV)
- Network information criterion (NIC)
- A Bayesian information criterion (ABIC)
- Vapnik’s measure (VM)

# Simulation Results



SIC outperforms other methods !!

# Conclusions



- We proposed an unbiased estimator of expected mean squared error (MSE) called subspace information criterion (SIC).
- Computer simulations showed that
  - SIC gives a very accurate estimate of MSE.
  - Optimal parameter values can be obtained by SIC.
  - SIC outperforms other methods.