Subspace Information Criterion for Image Restoration

— Mean Squared Error Estimator for Linear Filters

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We propose a method for determining parameter values appropriately.
Formulation

Hilbert space $H_1$

Original image $f$

Restored image $\hat{f}_w$

Hilbert space $H_2$

Observed image $g$

Degradation

$g = Af + n$

Filter

$\hat{f}_w = X_w g$

$w$: Parameter
Goal of This Talk

We want to determine the parameter value $w$ so that Mean Squared Error (MSE) is minimized.

$$\text{MSE} = \left\| \hat{f}_w - f \right\|^2$$

However, it is impossible to directly calculate MSE since $f$ is unknown.

We propose an estimator of MSE called the subspace information criterion (SIC), and determine $w$ so that SIC is minimized.
Bias / Variance Decomposition

\[ E \text{ MSE} = E \left\| \hat{f}_w - f \right\|^2 = \left\| Ef_w - f \right\|^2 + E \left\| \hat{f}_w - Ef_w \right\|^2 \]

- **Bias**: \[ \left\| Ef_w - f \right\|^2 \]
- **Variance**: \[ E \left\| \hat{f}_w - Ef_w \right\|^2 \]

\( E \): Expectation over noise
\( \hat{f}_w \): Restored image
\( f \): Original image
\( \sigma^2 \): Noise variance
\( X_w \): Restoration filter
\( X_w^* \): Adjoint of \( X_w \)
Key Idea for Estimating Bias

\( \hat{f}_u : \) Unbiased estimate of \( f \) \((E \hat{f}_u = f)\).

\[ \hat{f}_u = A^{-1}g \]

\[ E A^{-1}g = A^{-1}Af + E A^{-1}n = f \]

- \( E \): Expectation over noise
- \( f \): Original image
- \( g \): Degraded image
- \( A \): Degradation operator

\( \hat{f}_u \) is used for estimating bias!!
Bias Estimation

\[
\text{Bias} = \left\| E\hat{f}_w - f \right\|^2
\]

\[
= \left\| \hat{f}_w - \hat{f}_u \right\|^2 - 2\langle X_0 Af, X_0 n \rangle - \left\| X_0 n \right\|^2
\]

\[
\text{Bias} = \left\| \hat{f}_w - \hat{f}_u \right\|^2 - 0 - \sigma^2 \text{trace}(X_0 X_0^*)
\]

\[
E \text{ Bias} = \text{Bias}
\]

\[
X_0 = X_w - A^{-1}
\]

\[
f = E\hat{f}_u
\]

\[
X_w g \rightarrow \hat{f}_w \quad \text{Rough estimate} \quad \hat{f}_u \leftarrow A^{-1} g
\]
Subspace Information Criterion (SIC)

\[
SIC = \| \hat{f}_w - \hat{f}_u \|^2 - \sigma^2 \text{trace}(X_0 X_0^*) + \sigma^2 \text{trace}(X_w X_w^*)
\]

**Bias estimator**

**Variance**

\[
X_0 = X_w - A^{-1}
\]

\[
\sigma^2 : \text{Noise variance}
\]

SIC is an unbiased estimator of expected MSE:

\[
E \ SIC = E \ MSE
\]

\[
\text{MSE} = \| \hat{f}_w - f \|^2
\]
Simulation Setting

- Degradation operator $A$
  
  $f(x, y) \rightarrow [Af](x, y) = a_x f(x, y)$

- Noise $n(x, y) \sim N(0, \sigma^2)$
  
  $\sigma^2 = 900$

- Filter $X = X'A^{-1}$
  
  $X'$: Moving-average filter
Moving-Average Filter

\[ \hat{f}(x, y) = \sum_{i,j=-W}^{W} w_{ij} g(x-i, y-j) \]

Parameter: Window size \( W \)
Weight pattern \( \{ w_{ij} \} \)
Parameter candidates

- Window size $W = 0,1,\ldots,5$
- Weight pattern $\{w_{ij}\}$

(a) Rhombus  (b) Pyramid  (c) Gauss
Simulation Results: Lena

SIC: Gauss \((W=1)\), MSE=99

OPT: Gauss \((W=1)\)

\[ \hat{g} \]
Simulation Results: Peppers

Simulation Results: Peppers

MSE = 5572

SIC: Rhombus ($W=2$), MSE = 99

OPT: Rhombus ($W=2$)
Simulation Results: Girl

MSE = 3217

SIC: Rhombus ($W=2$), MSE = 77

OPT: Rhombus ($W=2$)
Simulation Setting (2)

- Degradation operator $A$

$$f(x, y) \xrightarrow{A} [Af](x, y) = \frac{1}{15} \sum_{i=-7}^{7} f(x - i, y)$$

- Noise $n(x, y) \sim N(0, \sigma^2)$

$$\sigma^2 = 16$$

- Filter: Regularization filter $X_\alpha$

$\alpha$: Regularization parameter selected from $\{10^{-5}, 10^{-4}, \ldots, 10^3\}$
Compared Methods

- Subspace information criterion (SIC)
- Mallows’s $C_L$ ($C_L$)
- Leave-one-out cross-validation (CV)
- Network information criterion (NIC)
- A Bayesian information criterion (ABIC)
- Vapnik’s measure (VM)
Simulation Results

SIC outperforms other methods!!
Conclusions

- We proposed an unbiased estimator of expected mean squared error (MSE) called subspace information criterion (SIC).

- Computer simulations showed that
  - SIC gives a very accurate estimate of MSE.
  - Optimal parameter values can be obtained by SIC.
  - SIC outperforms other methods.